

AD-A169 288

PREDICTING TRANSFORMS OF STABLE NOISE AND OTHER  
GAUSSIAN MIXTURES(U) NORTH CAROLINA UNIV AT CHAPEL HILL  
CENTER FOR STOCHASTIC PROC. R LEPAJE JUL 87 TR-193

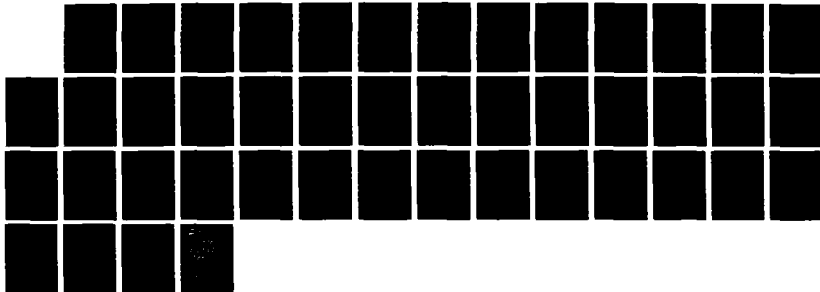
1/1

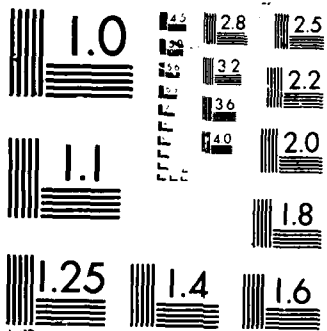
UNCLASSIFIED

AFOSR-TR-87-1888 F49628-85-C-8144

F/G 12/3

NL





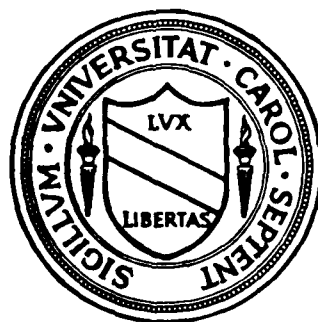
DTIC FILE COPY  
AFOSR-TR- 87-1808

2

# CENTER FOR STOCHASTIC PROCESSES

AD-A189 280

Department of Statistics  
University of North Carolina  
Chapel Hill, North Carolina



PREDICTING TRANSFORMS OF STABLE NOISE  
AND  
OTHER GAUSSIAN MIXTURES

by  
Raoul LePage

DTIC  
ELECTE  
JAN 14 1988  
S H D

Technical Report No. 193

July 1987

DISTRIBUTION STATEMENT A

Approved for public release

# REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3 DISTRIBUTION/AVAILABILITY OF REPORT  Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S)  Technical Report No. 193			5 MONITORING ORGANIZATION REPORT NUMBER(S) <b>AFOSR-TR- 87-1808</b>		
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION  AFOSR/NM		
6c. ADDRESS (City, State, and ZIP Code) Statistics Dept. 321-A Phillips Hall 039-A Chapel Hill, NC 27514			7b ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) NM	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR No. F49620 85C 0144.		
8c. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448			10 SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. H5
11 TITLE (Include Security Classification) Predicting transforms of stable noise and other Gaussian mixtures					
12. PERSONAL AUTHOR(S) LePage, R.					
13a. TYPE OF REPORT Preprint		13b. TIME COVERED FROM 9/85 TO 9/30/86		14. DATE OF REPORT (Year, Month, Day) July 1987	
15 PAGE COUNT 16 + second report					
16. SUPPLEMENTARY NOTATION N/A					
17. COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number) Key Words & Phrases:  N/A		
FIELD	GROUP	SUB-GROUP			
19 ABSTRACT (Continue on reverse if necessary and identify by block number)  Stationary stable processes that are Fourier transforms of  symmetric stable independent increments processes are shown to have  a.s. finite conditional expectation of $X_t$ given $X_s$ and conditional variance of $X_t$ given $X_{t-\delta}$ , $X_{t-2\delta}$ . The associated conditional expectation predictors are nonlinear in $\{X_s, s < t\}$ but are mixtures  of predictors of the usual type based on the Gaussian model.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21 ABSTRACT SECURITY CLASSIFICATION Unclassified/unlimited		
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj. Brian Woodruff			22b TELEPHONE (Include Area Code) (202) 767-5028		22c OFFICE SYMBOL NM

PREDICTING TRANSFORMS OF STABLE NOISE

AND

OTHER GAUSSIAN MIXTURES

by

Raoul LePage\*

Department of Statistics and Probability

Michigan State University, E. Lansing, MI 48824

\* Research partially supported by the Office of Naval Research

under grant USN N00014-85-K-0150 and Air Force Office of Scientific Research For

Research No. F49620 85 C 0144.



NTIS GRA&I

DTIC TAB

Unannounced

Justified

By

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

1/1/86

A-1

# ABSTRACT

Stationary stable processes that are Fourier transforms of symmetric stable independent increments processes are shown to have a.s. finite conditional expectation of  $X_t$  given  $X_s$  and conditional variance of  $X_t$  given  $X_{t-\delta}$ ,  $X_{t-2\delta}$ . The associated conditional expectation predictors are nonlinear in  $\{X_s, s < t\}$  but are mixtures of predictors of the usual type based on the Gaussian model.

## INTRODUCTION

The work presented here is part of a program extending the classical theories of prediction, detection, and smoothing of signals to encompass models in which Gaussian noise is replaced by symmetric  $\alpha$ -stable (SaS) noise, where  $0 < \alpha \leq 2$  is the index of stability. This is consistent with the full implications of the theory of errors (Levy, 1925) and includes Gaussian noise as the case  $\alpha = 2$ .

It is important to realise that Gaussian noise ( $\alpha = 2$ ) results in the frequency domain models being identical with the time domain models, by Bochner's Theorem (Doob, 1953). But the same is not true for  $\alpha < 2$ . That is, with the notable exception of  $\alpha = 2$ , the class FT-SaS of processes which are Fourier transforms of SaS noise (e.g. wave motions) will not be the same as the class of processes that are the outputs of linear systems, such as ARMA models, driven by SaS noise (Cline and Brockwell, 1985), (Makagon and Mandrekar, 1987).

Substantial progress has been made on the prediction problem for processes of the FT-SaS type, i.e. stationary processes  $X_t = \int e^{it\lambda} Z(d\lambda)$  where  $Z$  is an independent increments SaS process. In the course of this work a number of basic methods have been

developed which also have application to processes of the latter type, including ARMA models.

To understand the basic approach of this paper it is important to know that  $2n$  i.i.d. S $\alpha$ S random vectors tend to separate out like  $\pm j^{-1/\alpha}$ ,  $1 \leq j \leq n$  (LePage, Woodroffe, and Zinn, 1981), and are in general representable as mixtures of Gaussian distributions with differing covariances (LePage, 1980, appended to this report). This allows us to think of i.i.d. stable noise as inhomogeneous Gaussian noise with random  $\alpha/2$  stable covariances. As might be inferred from remarks made above, the consequences of this inhomogeneity when noise is in the form of wave amplitudes differ greatly from the effects of noise entering in the form of terms driving difference equations.

In recent work (Makagon and Mandrekar, 1987) there is defined the concept of a generalized spectrum for any strictly stationary stable process. In case the spectrum is given by independently scattered measure, as above, they have shown that one can obtain the linear analysis of such signals by methods analogous to those used for Gaussian processes. This extends (Cambanis and Soltani, 1982) to the case  $\alpha \leq 1$  by a more general method.

However, linear prediction is not in general as good as conditional expectation prediction. The latter has not been much



studied for stable processes because of the fact that  $E |X_t|$  is infinite for  $\alpha \leq 1$ . Surprisingly, by exploiting (LePage, 1980) we will prove for FT-SaS processes  $\{X_t\}$  that for every  $\alpha$  the conditional expectations  $E(|X_t| \mid X_s)$  and  $E(X_t^2 \mid X_{t-\delta}, X_{t-2\delta})$  are almost surely finite. This is surprising in view of the fact that for  $\alpha \leq 1$  the expectations  $E |X_t|$  are infinite, and for all  $\alpha < 2$  the expectations  $E X_t^2$  are infinite.

As a consequence of these conditional moments existing, the conditional expectation predictor of  $X_t$ , regardless of the number of predicting variables, is well-defined and optimal for conditional mean squared error, provided the predicting variables include two time points of the form  $t-\delta, t-2\delta$  for some  $\delta \neq 0$ . By combining these results with (Cambanis and LePage 1987) it can be shown that for  $\alpha < 2$  the conditional expectation predictor  $E(X_t \mid X_{t-\delta}, \delta_0 \leq \delta \leq L)$  is asymptotically consistent for  $X_t$  as  $\delta \rightarrow 0$  (excluding 0 in the discrete case) and  $L \rightarrow \infty$ , whereas the linear predictor is not.

Since the above conditional expectations turn out to be a-posteriori averages of Gaussian conditional expectations computed for various covariances, these non-linear conditional expectation predictors are a smoothing of Hilbert space methods and are in fact Bayesian predictors for a naturally occurring a-priori distribution intrinsic to FT-SaS processes.

The new methods generally allow us to compute quantities previously thought to be undefined because stable r.v. lack certain moments. For example, the identity  $E(X_t | X_s) = X_s E \cos(\Lambda(t-s))$ , where  $\Lambda$  is a random sample from the normalized spectral distribution function was known for  $1 < \alpha \leq 2$  (Kanter and Steiger, 1974). Corollary 2.1.2 below proves this result for  $0 < \alpha \leq 2$  by a totally new direct calculation which does not require existence of the unconditional expectation and bypasses differentiation of the characteristic function altogether. Certain other conditional and unconditional integrals can be directly calculated by the same method, including the integer moments of the characteristic function of the process conditioned on the invariant sigma algebra (Cambanis and LePage, 1987).

As mentioned previously, this work is based on a representation of FT-SoS processes as mixtures of stationary Gaussian processes (with randomly chosen covariance function  $\theta$ ) due to (LePage, 1980). The new observation, specialized to the case  $\mathcal{F} = \sigma\{X_{-1}, X_0\}$  and  $X = X_1$ , is that the conditional density of  $(X_1, X_0)$ , given the sigma field generated by the covariance function  $\theta$ , cancels terms in the conditional expected squared error of prediction. This forces convergence of the conditional expectation  $E^{\mathcal{F}}(X - E^{\mathcal{F}}X)^2$ . Such integrals, including  $E^{\mathcal{F}}X$ , are computed as mixtures, on  $\theta$ , of

$\theta$ -specific Gaussian integrals. The indicated predictors are thus conditional mixtures of  $\theta$ -specific Gaussian linear predictors.

### STABLE PROCESSES

The log characteristic function of any stationary Gaussian random functions having a continuous covariance takes the form

$$\log E \exp i \sum_{k=1}^n r_k X_{t_k} = -(\sigma^2/2) E |\sum_{k=1}^n r_k \exp i t_k \Lambda_1|^2, \quad (1.1)$$

where  $\sigma^2 > 0$  and  $\Lambda_1$  is a random variable whose probability distribution function is the normalized spectral distribution function. Stable analogues of these laws may be obtained by replacing the exponent 2 in (1.1) by a number  $\alpha$  in the range  $0 < \alpha < 2$ . In (LePage, 1980) it was proved that the resulting characteristic functions are precisely those of the class of FT-S $\alpha$ S processes. The following construction for such processes was given. For each  $0 < \alpha < 2$  define r.v.  $\{X(t), t \in \mathbb{R}\}$  by

$$X_t = \sum_{j=1}^{\infty} \cos(\Lambda_j t + \Theta_j) Y_j \Gamma_j^{-1/\alpha}. \quad (1.2)$$

In (1.2), which converges a.s. for each  $t$ , the sequence of r.v.  $\{\Lambda_j\}$  (which we denote by  $\Lambda$  for brevity) are i.i.d. from any distribution on  $\mathbb{R}$ ;  $\Theta$  are i.i.d. uniformly distributed on the interval  $[-\pi, \pi]$ ;  $Y$  are i.i.d. with  $E|Y_1|^\alpha$  finite;  $\Gamma$  are the consecutive arrival times of a homogeneous Poisson process with unit intensity function on the time domain  $\mathbb{R}^+$ ; and the sequences of r.v.  $\Lambda, \Theta, Y, \Gamma$  are mutually independent.

From (LePage, 1980, Theorem 7.4) the series (1.2) converges almost surely for each  $t \in \mathbb{R}$  and  $0 < \alpha < 2$ , and the r.v. defined by the left side of (1.2) have log characteristic function

$$2^{-\alpha} B(\alpha) C(\alpha) E |Y_1|^\alpha = E \left| \sum_{k=1}^n r_k \exp i t_k \Lambda_k \right|^\alpha, \quad (1.3)$$

where  $B(\alpha) = \alpha \int_0^\infty (\cos(r)-1) \frac{dr}{r^{1+\alpha}}$ , and  $C(\alpha) = \int_{-\pi}^{+\pi} |1+e^{i\eta}|^\alpha d\eta$ .

From (1.3) it may be seen that the law of  $X$  depends on the law of  $Y_1$  only through the  $\alpha$ -th absolute moment. Taking  $\{Y_j\}$  to be Rayleigh distributed, equivalent to letting  $\{Y_j\}$  be complex Gaussian and taking the real part of  $\{X(t), t \in \mathbb{R}\}$ , yields a process  $X$  which is conditionally Gaussian and stationary given the sequences  $\Lambda, \Gamma$ . The process given by the infinite series (1.1) may be written

$$X_t = \int_{\mathbb{R}} e^{i\lambda t} Z(d\lambda). \quad (1.4)$$

where  $Z$  is, conditionally on  $\Lambda, \Gamma$ , a Gaussian orthogonal random set function supported on the sequence of frequencies  $\Lambda$  and  $-\Lambda$ , with

$Z(\{-\Lambda_k\}) = \overline{Z(\{\Lambda_k\})}$  and  $Z(\{\Lambda_k\})$  given by

$$= \left(\frac{1}{2}\right) \sum_{j=1}^\infty \Gamma_j^{-1/\alpha} Y_j (I(\Lambda_j = \Lambda_k) e^{i\Theta_j} + I(-\Lambda_j = \Lambda_k) e^{-i\Theta_j}), \quad k \geq 1. \quad (1.5)$$

From (1.2) or (1.5) it is seen that the conditional covariance function of  $X$  given  $\Lambda, \Gamma$ , is given by

$$\theta(t) = \sum_{j=1}^\infty \Gamma_j^{-2/\alpha} \cos(\Lambda_j t)/2 \quad t \in \mathbb{R}. \quad (1.6)$$

That is,  $\{X(t), t \in \mathbb{R}\}$  is conditionally Gaussian and stationary with covariance (1.6) given  $\Lambda, \Gamma$ .

## 2. EXISTENCE OF CONDITIONAL MOMENTS

We require the formula  $E^{\mathcal{F}} \eta = \int \pi(d\theta | \mathcal{F}) E_{\theta}^{\mathcal{F}} \eta$ , where  $\eta$  is a non-negative polynomial in the coordinate r.v. of the random function  $X$ ,  $\mathcal{F}$  is the  $\sigma$ -field generated by a finite number of the coordinate r.v.,  $E_{\theta}^{\mathcal{F}} \eta$  is the  $E_{\theta}$ -conditional expectation given  $\mathcal{F}$ , and  $\pi(d\theta | \mathcal{F})$  is the regular conditional probability distribution of the finite number of coordinate values of  $\theta$  which appear in  $E_{\theta}^{\mathcal{F}} \eta$ . A proof may be given by martingale methods from the corresponding result for the discrete case which is  $P(A|D) = \sum_{\theta} P(\theta A D) / P(D) = \sum_{\theta} P(\theta | D) P_{\theta}(A|D)$ .

Theorem 2.1. Let  $X$  be a random function which is a mixture of zero-mean Gaussian stationary processes. For real numbers  $s, t$  the conditional expectation  $E^{\mathcal{F}} |X_t|$  is almost surely finite, where  $\mathcal{F}$  is a  $\sigma$ -algebra with respect to which  $X_s$  is measurable.

Proof. It is enough to prove that  $E^{X_0} |X_1|$  is finite a.s. If  $\theta(0) = 0$ , then  $X_0 = X_1 = 0$  almost surely. For each fixed  $\theta$  with  $\theta(0) > 0$ , the conditional distribution of  $X_1$  given  $X_0$  is normal with mean  $\mu(\theta, 0, 1) = E_{\theta}^{X_0} X_1 = (\theta(1)/\theta(0))X_0$ , and variance  $\sigma^2(\theta, 0, 1) = \theta(0)(1 - (\theta(1)/\theta(0))^2)$ . Then,

$$\begin{aligned} E^{X_0} |X(1)| &= \int_{(\theta(0) > 0)} E_{\theta}^{X_0} |X_1| \pi(d\theta | X_0) \\ &\leq \int_{(\theta(0) > 0)} (|\mu(\theta, 0, 1)| + \sigma(\theta, 0, 1)) \pi(d\theta | X_0). \end{aligned} \quad (2.2)$$

Since  $|\theta(1)| \leq |\theta(0)|$  a.s., the  $\mu$ -term of (2.2) is a.s. finitely integrable. Now use  $\pi(d\theta|X_0) = \pi(d\theta)\pi(X_0|\theta)\pi(X_0)$ , where  $\pi(r|\theta) = (2\pi)^{-1/2} \theta^{-1/2}(0) \exp(-r^2/2\theta(0))$ , from which it is seen that the  $\sigma$ -term of (2.2) is also a.s. finitely integrable.  $\square$

Corollary 2.1.1. If  $\mathcal{F}$  is a sigma algebra generated by a subset of the random variables  $\{X(t), t \in \mathbb{R}\}$ , then  $E^{\mathcal{F}} X_1 = \int \pi(d\theta|\mathcal{F}) E_{\theta}^{\mathcal{F}} X_1$ , which is a mixture of the  $\theta$ -specific Gaussian best linear predictors of  $X_1$  based on  $\mathcal{F}$ . In particular,  $E^{X_0} X_1 = E(\theta(1)/\theta(0)|X_0) X_0 = (\int_{(\theta(0)>0)} \pi(d\theta|X_0)(\theta(1)/\theta(0)) X_0$ .  $\square$

Corollary 2.1.2. (Cambanis-LePage). For a process of the form (1.2) with  $0 < \alpha \leq 2$ ,  $E(\theta(1)/\theta(0)|X_0) = E \cos(\Lambda_1)$ .

Proof. The case  $\alpha=2$  is obvious. For  $\alpha < 2$ , from (1.2) and (1.6)

$$X_0 = \sum_{j=1}^{\infty} \cos(\theta_j) \Gamma_j^{-1/2}$$

$$\theta(1)/\theta(0) = \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \cos(\lambda_j) / \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}.$$

Since  $\Lambda, \theta, \Gamma$  are independent and  $X_0$  is measurable  $\sigma\{\theta, \Gamma\}$ ,

$$E^{X_0} (\theta(1)/\theta(0)) = E^{X_0} E(\theta(1)/\theta(0)|\theta, \Gamma)$$

$$= \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} E \cos(\Lambda_j) / \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}$$

$$= E \cos(\Lambda_1). \quad \square$$

Since no special role is played by times zero and one, the formula

$E[\theta(t)/\theta(0)|X(s)] = E \cos(\Lambda_1(t-s))$  follows at once. This result extends (Kanter and Steiger, 1974) to the case  $\alpha \leq 1$ .

Proposition 1. There exist FT-SaS processes for which the

conditional mean squared error  $E^{X_0} (X_1 - E^{X_0} X_1)^2$  is infinite.

Proof. Let  $H(X_0) = \int (\theta(1)/\theta(0)) \pi(d\theta|X_0)$ . That is,  $E^{X_0} X_1 = H(X_0)$

$X_0$ . Then

$$\begin{aligned} E^{X_0} (X_1 - E^{X_0} X_1)^2 &= \int_{(\theta(0) > 0)} \theta(0) (1 - (\theta(1)/\theta(0)))^2 \pi(d\theta|X_0) \\ &+ \int_{(\theta(0) < 0)} X_0^2 (H(X_0) - (\theta(1)/\theta(0)))^2 \pi(d\theta|X_0). \end{aligned} \quad (2.3)$$

The second term on the right of (2.3) is integrable. The first term on the right may be handled using

$$\begin{aligned} \pi(d\theta|X_0) &= \pi(X_0|\theta) \pi(d\theta)/\pi(X_0) \\ &= (2\pi\theta(0))^{-1/2} e^{-X_0^2/2\theta(0)} \pi(d\theta)/\pi(X_0). \end{aligned} \quad (2.4)$$

From (1.6) it is seen that  $\theta(0)$  possesses a stable distribution of index  $\alpha/2$  under  $\pi(d\theta)$ . Taking  $\Lambda_1$  with support in the two point set  $\{-\pi/2, \pi/2\}$  ensures  $\theta(1)=0$  a.s., in which case the integrand of (2.3) is, by (2.4), asymptotically of order  $r^{-(\alpha+1)/2}$  at infinity.

This fails to be integrable if  $\alpha \leq 1$ .  $\square$

On the other hand, the conditional expected squared error of prediction given  $\mathcal{F} = \sigma\{X_{-1}, X_0\}$  is a.s. finite.

Theorem 2.2. Let  $X$  be a random function of the form (1.2). For real numbers  $r, s, t$  with  $|r-s| = |t-s|$  the conditional expectation  $E^{\mathcal{F}} (X_t - E^{\mathcal{F}} X_t)^2$  is almost surely finite if  $\mathcal{F}$  is a  $\sigma$ -algebra with respect to which  $X_r$  and  $X_s$  are measurable.

Proof. Replace  $\pi(d\theta|X_0)$  by  $\pi(d\theta|X_{-1}, X_0) = \pi(d\theta|\mathcal{F})$  in (2.3), and denote by  $\rho$  the correlation  $\rho = \theta(1)/\theta(0)$ . Then,

$$\pi(d\theta|\mathcal{F}) \leq (\pi(d\theta)/\pi(X_{-1}, X_0))(2\pi)^{-1}\theta^{-1}(0)(1-\rho^2)^{-1}. \quad (2.5)$$

From theorem 2.1,

$$\begin{aligned} E^{\mathcal{F}}X^2(1) &= \int \pi(d\theta|\mathcal{F}) E_{\theta}^{\mathcal{F}}X_1^2 \\ &= \int \pi(d\theta|\mathcal{F}) [E_{\theta}^{\mathcal{F}}(X_1 - E_{\theta}^{\mathcal{F}}X_1)^2 + (E_{\theta}^{\mathcal{F}}X_1)^2] \end{aligned} \quad (2.6)$$

The first term in the brackets of (2.6) is less than or equal to  $\theta(0)(1-\rho^2)$ , and is therefore  $\pi(d\theta)$ -integrable when multiplied by the Jacobian from (2.5). The second term in the brackets of (2.6) may, on  $(\rho^2 \neq 1)$  be written

$$[\rho X_0 + \frac{\theta(2) - \rho\theta(1)}{\theta(0)(1-\rho^2)} (X_{-1} - \rho X_0)]^2. \quad (2.7)$$

The first component in (2.7) is  $\rho X_0$  whose square is integrable by  $\pi((d\theta|\mathcal{F}))$ . It is therefore enough to bound

$$D(\theta) = \left[ \frac{\theta(2) - \rho\theta(1)}{\theta(0)(1-\rho^2)} \right] \quad (2.8)$$

Let  $|| \quad ||$  denote  $L_2$  norm with respect to the zero-mean stationary Gaussian process with covariance function  $\theta$ . Then

$$\begin{aligned} \theta(0) &= ||X_1||^2 \geq ||\rho X_0||^2 + ||D(\theta)(X_{-1} - \rho X_0)||^2 \\ &= \rho^2 \theta(0) + D^2(\theta) \theta(0) (1-\rho^2). \end{aligned}$$

Therefore  $D^2(\theta) \leq 1$  on  $\rho^2 \neq 1$ . But on  $\rho^2 = 1$  (2.7) is replaced by  $X_0$ .

Thus (2.5) is a.s. finite. Note that the Cauchy-Schwartz bound on (2.8) is inadequate.  $\square$

### 3. FUTURE DIRECTIONS



The ergodic properties of  $\{X_t\}$  are now completely understood. In (Cambanis and LePage, 1987), subject to mild conditions on the spectral distribution, a complete characterization of the invariant sigma field of the process  $\{X_t\}$  is obtained in terms of amplitude and frequency variables measurable with respect to the remote past. The non-ergodic component is identified with i.i.d. uniformly distributed phase variables.

Using the above results, it follows that under rather weak conditions the predictors  $E(X_t \mid X_{t-\delta}, \delta_0 \leq \delta \leq L)$  converge almost surely to  $X_t$  (and conditionally in mean square given the random spectral amplitudes) as  $L \rightarrow \infty$ ,  $\delta_0 \rightarrow 0$ . Thus, the action of SaS noise in wave amplitudes is to make prediction ultimately more perfect than would be the case for Gaussian noise. Since by a result of (Makagon and Mandrekar, 1987) linear prediction optimized for the given FT SaS distribution is only consistent for  $X_t$  under very exceptional conditions on the spectral measure, it follows that linear prediction is inferior to conditional expectation prediction for  $\alpha < 2$ . Such results generalize easily to the multiparameter and  $\dim > 1$  cases, and will apply to certain types of spatial processes which are Fourier transforms.

By contrast with the above, linear prediction for linear system SaS processes, such as ARMA(p,q) models driven by i.i.d. SaS random

vectors, is intrinsically interesting due to the way in which the noise enters additively. The conditional expectations  $E(|X_t| \mid X_s, s < t)$  are a.s. infinite except in trivial cases due to  $E(|\epsilon_t| \mid X_s, s < t) = \infty$ , i.e. the conditional expectation of the independent error. By Theorem 2.1 it follows that such processes cannot be of the FT SoS type.

If it can be proved that  $E(|\epsilon_r| \mid X_s, s < t) < \infty$  a.s. for all  $r < t$ , then an obvious choice for non-linear prediction will be to predict the value zero for  $\epsilon_t$  and use the conditional expectation predictor on  $\epsilon_r$ ,  $r < t$ . This most promising approach is currently under study.

## REFERENCES

Cambanis, S. and LePage, R. (1987). Harmonizable stable processes.  
Tech. Rept., Center for Stochastic Processes, Univ. of North  
Carolina. (in preparation).

Cambanis, S. and Soltani, A.R. (1982). Prediction of stable  
processes: Spectral and moving average representations. Tech. Rept.  
No. 11, Center for Stochastic Processes, Univ. of North Carolina.

Cline D. B. H. and Brockwell, P. J. (1985) Linear prediction of ARMA  
processes with infinite variance. Stochast. Proc and Applns. 19,  
281-289.

Doob, J. L. (1953) Stochastic Processes. John Wiley & Son Inc., New  
York.

Hosoya, Y. (1982). Harmonizable stable processes. Z. Wahr. und verw.  
Geb. 60, 517, 533.

Kanter, M. and Steiger, W.L., (1974). Regression and autoregression  
with infinite variance, Adv. Appl. Prob., 6, 768-783.

LePage, R. (1980). Multidimensional infinitely divisible variables and processes, Part I: Stable case. Tech. Rept. No. 292, Statistics Department, Stanford University.

LePage, R. (1981). Multidimensional infinitely divisible variables and processes, Part II. Lecture Notes in Mathematics, 860, 279-284.

LePage, R., Woodroffe, M., and Zinn, J. (1981). Convergence to a stable distribution via order statistics. Ann. Prob. 9, 624-632.

Levy, P. (1925). Calcul des Probabilites Gauthier-Villars, Paris.

#### Appendix

A reproduction of LePage, R. (1980). Multidimensional infinitely divisible variables and processes, Part I: Stable case. Tech Rept. No. 292, Statistics Department, Stanford University follows.

MULTIDIMENSIONAL INFINITELY DIVISIBLE VARIABLES

AND PROCESSES. PART I: STABLE CASE

By

Raoul LePage

Partially supported by National Science Foundation Grant MCS 78-26143,  
and the Office of Naval Research Contract N00014-76-C-0425.

# Abstract

Elementary series constructions, involving a Poisson process, are obtained for multidimensional stable variables and random functions. Symmetric stable laws are shown to be mixtures of Gaussian laws.

AMS 1970 subject classifications. Primary 60E07; Secondary 62E10

Key words and phrases. Stable, series representation, Poisson process.

MULTIDIMENSIONAL INFINITELY DIVISIBLE VARIABLES  
AND PROCESSES. PART I: STABLE CASE

By

Raoul LePage

1. Introduction.

Series decompositions, involving the arrival times of a Poisson process, have been given by Ferguson and Klass [1] for the non-Gaussian component of an arbitrary (real-valued) independent increments random function on the unit interval. LePage and Woodroffe and Zinn [4] have rediscovered a variant of this decomposition in connection with their study, via order statistics, of the limit distribution for self-normalized sums (e.g. Students'-t), when sampling from a distribution in the domain of attraction of an arbitrary stable law of index  $0 < \alpha \leq 2$ .

The present paper obtains a characterization of stable laws on spaces of dimension greater than one. This characterization is formally like that of Ferguson-Klass for dimension one, but with i.i.d. vector multipliers on the Poisson terms. The law of these coefficients may be chosen proportional to the Lévy measure, although this is not necessary. These results take a particularly elegant form in the case of symmetric stable laws, where something of a calculus is developed showing:

- (i) which Lévy measure associates with vector coefficients other than the aforementioned ones, (ii) what happens when independent stables are linearly combined as in a weighted sum, (iii) how to construct an arbitrary multidimensional independent-increment, symmetric, stable set

function, and (iv) how to construct an arbitrary harmonizable stationary symmetric stable random function having multidimensional domain and/or range. Symmetric stable laws are shown to be mixtures of Gaussian laws.

Partly because of the self-contained character of Kuelbs' paper [3], in which the characterization of the log-characteristic function of a stable law is extended to real separable Hilbert space, the Hilbert space level of generality has been chosen for this paper. Later extensions of Kuelbs' result to Banach and more general spaces support a corresponding generalization of these results. In addition to Kuelbs' result we need a method employed by Ferguson and Klass to transform certain dependent series into eventually identical independent ones. We also require standard results giving conditions under which an independent series in Hilbert space converges almost surely (e.g. [2], Theorem 5.3). The rest of the paper is basically self-contained and affords a surprisingly accessible and clear view of  $\alpha < 2$  stable laws, and random functions, based on elementary series constructions.

Part II of this paper will generalize these results to the infinitely divisible case.



## 2. Notation.

The following symbols and conventions will be in force throughout this paper.

(2.1)	$\sim$	"is asymptotic with"
	$\triangleq$	"equals by definition"
	$\stackrel{D}{=}$	"has the same distribution as"
	$\Rightarrow$	"converges in distribution to"
	$\alpha$	$0 < \alpha < 2$ , an index of stability
	$\{\Gamma_j, j \geq 1\}$	arrival times of a Poisson process with unit rate
	$H$	a real separable Hilbert space

The material of the next section is drawn from [4].

### 3. Motivation.

Limit theorems are not the subject of this paper. However, we should not proceed without benefit of the following example, which exposes some connections between  $\alpha < 2$  stable r.v. and the Poisson process.

Let  $\{\varepsilon_j, j \geq 1\}$  be independent of the sequence  $\Gamma$  and i.i.d. with  $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$ , and define  $G(x) \triangleq x^{-\alpha}$ ,  $\forall x \geq 1$ . Think of  $G(x) = P(|X| \geq x) \forall x \geq 1$ , where  $X$  is a r.v. symmetrically distributed about zero. We will construct particular r.v.  $X_1, \dots, X_n$ , i.i.d. as  $X$ , whose normalized sums converge in distribution to the symmetric stable law of index  $\alpha$ . To do this, use the arrival times of a Poisson process to generate uniform order statistics, apply  $G^{-1}$  to these, multiply by the signs  $\varepsilon$ , and permute. As constructed, the normalized sums will actually converge almost surely to  $\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha}$  (see (3.1) below), a series possessing the symmetric stable law of index  $\alpha$ . In fact, a direct proof of the stability of  $\sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha}$  follows easily from the observation that the arrival times of several (say  $K > 1$ ) independent unit rate Poisson processes (run simultaneously) constitute  $K^{-1}$  times the arrival times of a unit rate Poisson process. This argument works just as well for  $\varepsilon$  replaced by any vector sequence (provided the series converges) and suggests the multivariate extensions of sections 4 and 5.

For each  $n > 1$  let  $U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(n)}$  denote the order statistics of i.i.d. random variables  $U_1, \dots, U_n$  which are

uniformly distributed on  $[0,1]$ . Then for each fixed  $n > 1$ , letting

$$X_j \triangleq \varepsilon_j G^{-1}(U_j), \quad \forall j \geq 1,$$

$$\begin{aligned}
 (3.1) \quad n^{-1/\alpha} \sum_1^n \varepsilon_j X_j &\stackrel{D}{=} n^{-1/\alpha} \sum_1^n \varepsilon_j G^{-1}(U_{(j)}) \\
 &= n^{-1/\alpha} \sum_1^n \varepsilon_j U_{(j)}^{-1/\alpha} \\
 &\stackrel{D}{=} n^{-1/\alpha} \sum_1^n \varepsilon_j (\Gamma_j / \Gamma_{n+1})^{-1/\alpha} \\
 &= \underbrace{\sum_1^n \varepsilon_j \Gamma_j^{-1/\alpha}}_{\downarrow \text{a.s. } 1} \underbrace{(\Gamma_{n+1}/n)^{1/\alpha}}_{\downarrow \text{a.s. (SLLN)}} \\
 &\quad \sum_1^\infty \varepsilon_j \Gamma_j^{-1/\alpha} = 1
 \end{aligned}$$

It is convenient to refer to  $\varepsilon_j \Gamma_j$ ,  $j \geq 1$  as the residual order statistics, keeping in mind that the ordering is on decreasing absolute values.

The same example suggests an invariance principle (proved in [4]) for self-normed sums such as  $\sum_1^n X_j / \sqrt{\sum_1^n X_j^2}$  which, regardless of  $\alpha$ , converge in distribution to a limit law depending only on the stable attracting  $X_1$ . For the r.v. constructed above,  $\forall n \geq 1$ ,

---

1. Use  $\Gamma_j \sim j$  and  $\sum_1^\infty j^{-2/\alpha} < \infty$  a.s., and apply the 3-series theorem conditional on the sequence  $\Gamma$ .

$$(3.2) \quad \frac{\sum_1^n X_j}{\sqrt{\sum_1^n X_j^2}} \stackrel{D}{=} \frac{\sum_1^n \varepsilon_j \Gamma_j^{-1/\alpha}}{\sqrt{\sum_1^n \Gamma_j^{-2/\alpha}}} \xrightarrow{a.s.} \frac{\sum_1^\infty \varepsilon_j \Gamma_j^{-1/\alpha}}{\sqrt{\sum_1^\infty \Gamma_j^{-2/\alpha}}}$$

That is, the limit law of the t-statistic<sup>2</sup> is that of the t-statistic calculated on the residual order statistics (see also [6]).

Even the construction of stable independent-increments processes can be motivated by means of the same example. We restrict our attention to the homogeneous increments case. Let  $[T_j, j \geq 1] \stackrel{D}{=} [U_j, j \geq 1]$ , and suppose the sequences  $T, \varepsilon, \Gamma$  are mutually independent. The partial sum processes  $\sum_1^{[nt]} X_j$ ,  $0 \leq t \leq 1$ ,  $\forall n \geq 1$  can be effected by independent selections of  $X_1, \dots, X_n$  into subsets of sizes  $[nt]$  using multiplication by indicators:

$$(3.3) \quad I_1^{(n)}(t) \stackrel{\Delta}{=} I(T_1 \leq \frac{[nt]}{n})$$

$$I_j^{(n)}(t) \stackrel{\Delta}{=} I(T_j \leq \frac{[nt] - \sum_{i < j} I_i^{(n)}(t)}{n+1-j}), \quad \forall 1 \leq j \leq n.$$

Then for each  $n \geq 1$ ,

$$(3.4) \quad \{n^{-1/\alpha} \sum_1^{[nt]} X_j, t \in [0,1]\} \stackrel{D}{=} \{\sum_1^n I_j^{(n)}(t) \varepsilon_j \Gamma_j^{-1/\alpha} (\Gamma_{n+1}/n)^{1/\alpha}, t \in [0,1]\}$$

$\downarrow$  a.s. in  $D[0,1]$

$$\{\sum_1^\infty I(T_j \leq t) \varepsilon_j \Gamma_j^{-1/\alpha}, t \in [0,1]\}.$$

Details of this argument are unpublished.

<sup>2</sup>. The square of this t-statistic is simply related to the square of Students'-t, and both have the same limit law.

#### 4. Stable Laws on H.

Suppose  $\{X_j, j \geq 1\}$  are i.i.d. random vectors in a real separable Hilbert space  $H$  and that the sequences  $X, \Gamma$  are independent. For each  $n \geq 1$  denote by  $K_n$  the number of arrival times  $\Gamma$  in the interval  $[0, a_n]$ ,  $a_n \triangleq \sum_{j=1}^n j^{-1}$ . This choice of  $a_n$  is from [1]. Its advantages will be apparent in what follows.

Remark. Sums of the kind  $\sum_{j=1}^{K_n}$  are for each  $n \geq 1$  defined to zero on the event  $K_n = 0$ . Use  $(\cdot, \cdot)$ ,  $\|\cdot\|$ , to denote  $H$  inner product and norm.

For each  $n \geq 1$ ,  $x \in H$ ,  $c > 0$ , (see also [1], pg. 1639),

$$\begin{aligned}
 (4.1) \quad E e^{i(x, c \sum_{j=1}^{K_n} X_j \Gamma_j^{-1/\alpha})} &= E E_n^K e^{i c \sum_{j=1}^{K_n} (x, X_j) \Gamma_j^{-1/\alpha}} \\
 &= E (a_n^{-1} \int_0^{a_n} E e^{i c (x, X_1) t^{-1/\alpha}} dt)^{K_n} \\
 &= E \int_0^{a_n} (e^{i c (x, X_1) t^{-1/\alpha}} - 1) dt \\
 &= E e^{c^\alpha \alpha \int_{c a_n^{-1/\alpha}}^\infty (e^{i(x, X_1) r} - 1) \frac{dr}{r^{1+\alpha}}}
 \end{aligned}$$

Kuelbs ([3], lemma 2.2) has proved that the log-characteristic function of a (non-Gaussian) stable probability measure on  $H$  is necessarily of the following form, for a unique  $S \in H$  and finite Borel measure  $\sigma$  on  $S \triangleq \{x \in H \mid \|x\| = 1\}$ ,

$$(4.2) \quad i(x, \beta) + \int_S \int_0^\infty (e^{i(x,s)r} - 1 - \frac{i(x,s)r}{1+r^2}) \frac{dr}{r^{1+\alpha}} \sigma(ds), \quad \forall x \in H.$$

It is convenient to refer to:

(4.3) expression (4.2) with  $\beta=0$ .

Define  $\delta \triangleq \sigma(S)$ ,  $\mu \triangleq \sigma/\delta$ , and

$$\beta_n \triangleq \int_S \int_{\alpha^{-1/\alpha} \delta^{-1/\alpha} a_n^{-1/\alpha}}^\infty \frac{sr}{1+r^2} \frac{dr}{r^{1+\alpha}} \sigma(ds), \quad \forall n \geq 1.$$

Lemma 4.4 If (4.3) is the characteristic function of a stable law on  $H$  and if the sequence  $\{X_j, j \geq 1\}$  is i.i.d.  $\mu$  and independent of the sequence  $\Gamma$ , define

$$X^{(n)} \triangleq \alpha^{-1/\alpha} \delta^{1/\alpha} \sum_{j=1}^{K_n} X_j \Gamma_j^{-1/\alpha} - S_n, \quad \forall n \geq 1.$$

Then  $\forall x \in H$ ,  $E \exp i(x, X_n)$  converges to (4.3).

Remark. This result is not altogether satisfactory since convergence of the series  $\{X^{(n)}, n \geq 1\}$  is through stochastic times  $\{K_n, n \geq 1\}$ , and is not yet a.s. in  $H$ . These defects are remedied in Theorem 4.8 below.

Proof. For each  $x \in H$ ,  $n \geq 1$ , by (4.1),

$$\begin{aligned}
(4.5) \quad \log_e E e^{i(x, X^{(n)})} &= -i(x, \beta_n) + E \int_{\alpha^{-1/\alpha} \delta}^{\infty} \alpha^{-1/\alpha} a_n^{-1/\alpha} (e^{i(x, X_1)r} - 1) \frac{dr}{r^{1+\alpha}} \\
&= \int_s \int_{\alpha^{-1/\alpha} \delta}^{\infty} \alpha^{-1/\alpha} a_n^{-1/\alpha} (e^{i(x, s)r} - 1 - \frac{i(x, s)r}{1+r^2}) \frac{dr}{r^{1+\alpha}} \sigma(ds)
\end{aligned}$$

→ (4.3)      □

From ([1], lemma 2), we conclude that

$$\left\{ \sum_{K_n}^{K_{n+1}} X_j \Gamma_j^{-1/\alpha}, n \geq 1 \right\}$$

are independent. Furthermore, using an argument drawn from [4],  $\forall n > 1$ ,

$$(4.6) \quad E \left\| \sum_{K_n}^{K_{n+1}} X_j \Gamma_j^{-1/\alpha} \right\|^2 \leq a_n^{-2/\alpha} E(K_{n+1} - K_n)^2 \sim < \frac{2}{n(\log_e n)^{2/\alpha}},$$

which is summable in  $n$ . Therefore, :

$$\sum_1^K X_j \Gamma_j^{-1/\alpha} - E \sum_1^K X_j (\Gamma_j^{-1/\alpha} \wedge 1)$$

converges in probability in  $H$ . A short calculation gives

$$(4.7) \quad A_n(\alpha, \delta) \triangleq E \alpha^{-1/\alpha} \delta^{1/\alpha} \sum_{j=1}^K (\Gamma_j^{-1/\alpha} \wedge 1) - \delta \int_{\alpha^{-1/\alpha} \delta^{1/\alpha} a_n^{-1/\alpha}}^{\infty} \frac{r}{1+r^2} \frac{dr}{r^{1+\alpha}}$$

$$\stackrel{3.}{=} \alpha^{-1/\alpha} \delta^{1/\alpha} \int_0^{a_n} ((t^{-1/\alpha} \wedge 1) - \frac{t^{-1/\alpha}}{1+\alpha^{-2/\alpha} \delta^{2/\alpha} t^{-2/\alpha}}) dt$$

$$\rightarrow \text{finite limit} \stackrel{\Delta}{=} A(\alpha, \delta) \text{ as } n \rightarrow \infty.$$

Therefore  $X^{(n)}$  converges in probability.

Theorem 4.8. If (4.3) is the log characteristic function of a stable law on  $H$  then the series

$$\alpha^{-1/\alpha} \delta^{1/\alpha} \sum_{j=1}^n (X_j \Gamma_j^{-1/\alpha} - (EX_1)) \int_{j-1}^j \frac{t^{-1/\alpha}}{1+\alpha^{-2/\alpha} \delta^{2/\alpha} t^{-2/\alpha}} dt$$

converges a.s. in  $H$  to a random vector with log characteristic function (4.3).

Remark. Centering is not needed for the case  $\alpha < 1$ , nor is it needed for the symmetric case which will appear in [4].

Proof. Since  $X^{(n)}$  converges in probability and  $X^{(n)}$  is an independent series, we conclude by ([2], Theorem 5.3(6)) that  $X^{(n)}$  converges a.s. in  $H$ . Recall that with probability one  $\exists$  finite  $M$  such that  $(n \geq M) \Rightarrow (\exists \text{ smallest } N(n) \text{ with } n = K_{N(n)})$ . Then  $\forall n \geq m$ ,

3. The first term equals the first term above. For the second term use  $t = \alpha^{-1/\alpha} \delta^{1/\alpha} r^{-\alpha}$



$$(4.9) \quad X^{(N(n))} = \alpha^{-1/\alpha} \delta^{1/\alpha} \left\{ \sum_1^n X_j \Gamma_j^{-1/\alpha} - (EX_1) \int_0^{a_{N(n)}} (t^{-1/\alpha} \wedge 1) dt \right\} + \\ (EX_1) A_n(\alpha, \delta) + o(1) .$$

Since for  $n \geq M$ ,  $(n = K_{N(n)}) \Rightarrow (a_{N(n)} - 1 < \Gamma_n \leq a_{N(n)})$ ,

$$(4.10) \quad \left| \int_{\Gamma_n}^{a_{N(n)}} (t^{-1/\alpha} \wedge 1) dt \right| \leq (N(n))^{-1} \xrightarrow{\text{a.s.}} 0 .$$

By the law of the iterated logarithm, a.s. eventually as  $n \rightarrow \infty$

$$(4.11) \quad \left| \int_n^{\Gamma_n} (t^{-1/\alpha} \wedge 1) dt \right| = \int_n^{\Gamma_n} t^{-1/\alpha} dt \leq (\Gamma_n - n) (n + \Gamma_n)^{-1/\alpha} \\ \leq 2 \sqrt{n \log \log n} (n + o(n))^{-1/\alpha} \rightarrow 0 .$$

Therefore,

$$(4.12) \quad X^{(N(n))} = \alpha^{-1/\alpha} \delta^{1/\alpha} \left\{ \sum_1^n X_j \Gamma_j^{-1/\alpha} - (EX_1) \int_0^n (t^{-1/\alpha} \wedge 1) dt \right\} - \\ A(\alpha, \delta) (EX_1) + o(1) ,$$

which converges a.s. in  $H$  to a random vector with log characteristic function (4.3).  $\square$

Remark. The centerings used above also have an interpretation involving  $(EX_1) E(\Gamma_j^{-1/\alpha} \wedge 1)$ , which will not be given here.

## 5. Symmetric Case, Multiple Representations.

In this section we do not assume that  $X_1$  is distributed according to the measure  $\mu$ , or even restrict its distribution to  $S$ . Suppose  $X, \varepsilon, \Gamma$  are mutually independent sequences, with  $\varepsilon, \Gamma$  as in section 3, and  $\{X_j, j \geq 1\}$  i.i.d. in  $H$  with  $E\|X_1\|^\alpha < \infty$ .

Remark. Series of form  $\sum_{j=1}^n \varepsilon_j X_j \Gamma_j^{-1/\alpha}$  will be termed symmetric.

Lemma 5.1. The symmetric series  $\sum_{j=1}^n \varepsilon_j X_j \Gamma_j^{-1/\alpha}$  converges a.s. in  $H$  and the log characteristic function of its limit is  $E|(x, X_1)|^\alpha B(\alpha)$ ,  $\forall x \in H$ , where  $B(\alpha) \triangleq \int_0^\infty \alpha(\cos(r)-1) \frac{dr}{r^{1+\alpha}}$ .

Proof. The arguments needed are similar to those of theorem 4.8, but easier in the symmetric case. For each  $x \in H$ ,

$$\begin{aligned}
 (5.2) \quad \log_e E e^{i(x, \sum_{j=1}^K \varepsilon_j X_j \Gamma_j^{-1/\alpha})} &\stackrel{(4.1)}{=} \alpha E \int_{a_n^{-1/\alpha}}^\infty (e^{i\varepsilon_1 (x, X_1) r} - 1) \frac{dr}{r^{1+\alpha}} \\
 &= \alpha E \int_0^\infty (e^{i\varepsilon_1 r} - 1) I(|(x, X_1)| \leq r a_n^{1/\alpha}) (x, X_1)^\alpha \frac{dr}{r^{1+\alpha}} \\
 &= \alpha E \int_0^\infty (\cos(r)-1) I(|(x, X_1)| \leq r a_n^{1/\alpha}) (x, X_1)^\alpha \frac{dr}{r^{1+\alpha}} \\
 &\rightarrow \alpha \int_0^\infty (\cos(r)-1) \frac{dr}{r^{1+\alpha}} E (x, X_1)^\alpha.
 \end{aligned}$$

From ([3], corollary 2.1), the limit in (5.2) is the characteristic

function of a (symmetric) stable law on  $H$ . Since the sums

$\sum_{j=1}^{K_{n+1}} \varepsilon_j X_j \Gamma_j^{-1/\alpha}$  are independent and symmetric, we have from ([2], theorem 5.3(1)) that  $\sum_{j=1}^{K_n} \varepsilon_j X_j \Gamma_j^{-1/\alpha}$  converges a.s. in  $H$ . Since eventually a.s.  $K_{n+1} \leq K_n + 1$  as  $n \rightarrow \infty$ , we conclude  $\sum_{j=1}^n \varepsilon_j X_j \Gamma_j^{-1/\alpha}$  converges a.s. in  $H$ .  $\square$

Several series may represent the same stable law.

Theorem 5.2. If  $E \|X_1\|^\alpha < \infty$  then for every  $x \in H$ ,

$E |(x, X_1)|^\alpha = E \|X_1\|^\alpha E |(x, X_1^*)|^\alpha$  where  $X_1^*$  is distributed on  $S$  according to the measure:

$$P(X_1^* \in A) = E I\left(\frac{X_1}{\|X_1\|} \in A\right) \frac{\|X_1\|^\alpha}{E \|X_1\|^\alpha}.$$

Proof. For every  $x \in H$ ,

$$\begin{aligned} E |(x, X_1)|^\alpha &= E \left| \left(x, \frac{X_1}{\|X_1\|}\right) \right|^\alpha \|X_1\|^\alpha \\ &= E \|X_1\|^\alpha E |(x, X_1^*)|^\alpha. \quad \square \end{aligned}$$

As an example of the above, every symmetric stable law on  $H$  has a construction  $\sum_{j=1}^\infty Z_j X_j \Gamma_j^{-1/\alpha} / E(|Z_1|^\alpha)^{1/\alpha}$  in terms of an independent sequence  $Z$  of i.i.d. standard normal r.v.. Conditional on the sequences  $\Gamma$  and  $X$  the symmetric stable is Gaussian. That is, symmetric stable laws are particular mixtures of Gaussian laws with zero means and differing covariance kernels. The latter will not in general differ only by scale, though this is necessarily true for  $H = R_1$ .

## 6. Symmetric Stable Random Set Functions.

Let  $\{\tau_j, j \geq 1\}$  be i.i.d. taking values in a measurable space with measurable sets generically denoted  $A$ . Suppose  $\tau, \varepsilon, X, T$  are mutually independent, where the latter three sequences are as in section 5. Define

$$(6.1) \quad X(A) \stackrel{\Delta}{=} \sum_{j=1}^{\infty} I(\tau_j \in A) X_j \varepsilon_j r_j^{-1/\alpha}, \quad \forall A.$$

Theorem 6.2. The series (6.1) is a.s. convergent for each  $A$ , is jointly symmetric stable for finitely many  $A$  at a time, and  $X(A_1), \dots, X(A_n)$  are for each  $n \geq 1$  mutually independent if  $A_1, \dots, A_n$  are mutually disjoint.

Proof. For each  $n \geq 1$ ,  $x \in H$ , real numbers  $r_1, \dots, r_n$ , and measurable sets  $A_1, \dots, A_n$ ,

$$(6.3) \quad \log_e E e^{i(x, \sum_{k=1}^n r_k X(A_k))} = \log_e E e^{i \sum_{j=1}^{\infty} (\sum_{k=1}^n r_k I(\tau_j \in A_k)) (x, X_j) \varepsilon_j r_j^{-1/\alpha}}$$

$$(5.1) \\ = B(\alpha) E^1(x, X_1)^{\alpha} E \left( \sum_{k=1}^n r_k I(\tau_1 \in A_k) \right)^{1/\alpha}.$$

If  $A_1, \dots, A_n$  are mutually disjoint then

$$(6.4) \quad E \left( \sum_{k=1}^n r_k I(\tau_1 \in A_k) \right)^{1/\alpha} = \sum_{k=1}^n r_k^{1/\alpha} P(\tau_1 \in A_k). \quad \square$$

Remark. The simplicity of this construction is interesting, as is the way in which  $\alpha$ -dependence, dimensional structure, and functional dependence are identified with mutually independent coefficient sequences  $r^{-1/\alpha}$ ,  $X$ ,  $Z$ .

Remark. Schilder [8] and Kuelbs [3] have explored a representation of multidimensional symmetric stable r.v. by means of a stochastic integral with respect to a one-dimensional stable independent increments process. Theorem (4.8) and lemma (5.1) sharpen and extend such representations by connecting them with the Ferguson-Klass representation, making explicit the choice of coefficients required to obtain each stable law<sup>4</sup>, and establishing  $H$  convergence of the indicated series.

Remark. Suppose  $K \geq 1$  and  $Y_k = \sum_{j=1}^{\infty} \xi_{kj} r_{kj}^{-1/\alpha}$ ,  $1 \leq k \leq K$ , are independently constructed (as per (5.1)) symmetric stable r.v. taking values in  $H$ . Then for an arbitrary choice of real numbers  $r_1, \dots, r_K$  the sum  $\sum_{k=1}^K r_k Y_k$  is representable  $\sum_{j=1}^{\infty} \xi_j r_j^{-1/\alpha}$  where  $\xi_j$ ,  $j \geq 1$ , are i.i.d. and  $\xi_1 \stackrel{D}{=} \text{an equiprobable random selection from } K^{1/\alpha} r_1^{1/\alpha}, \dots, K^{1/\alpha} r_K^{1/\alpha}$ . This uses the property (discussed in section 3) of  $K$  Poisson processes run simultaneously.

<sup>4</sup>. Including the non-symmetric stable laws.

## 7. Harmonisable Stationary Symmetric Stable Random Functions.

Basically, we seek to construct the stable analogues of Gaussian stationary random functions having a harmonic decomposition. The characteristic function of such a Gaussian random function involves

$$E \left| \sum_{k=1}^n r_k e^{i(\Lambda_1, t_k)_0} \right|^2,$$

where  $t$  is generic for a point of the domain, and  $(\Lambda_1)_0$  is a random linear function on the domain. The stable analogues of these Gaussian random functions have characteristic functions that employ an  $\frac{1}{\alpha}$ -power in this integral instead of the 2, but are otherwise the same. Define  $\forall t$ ,

$$(7.1) \quad X(t) \equiv \sum_{j=1}^{\infty} \cos((\Lambda_j, t)_0 + \Theta_j) X_j \varepsilon_j r_j^{-1/\alpha},$$

where  $\Lambda_j$  are i.i.d.,  $\Theta_j$  are i.i.d. uniforms on  $[-\pi, \pi]$ ,  $\{\varepsilon, X, \Gamma\}$  are as in section 6, and  $\Lambda_j, \Theta_j, \varepsilon_j, X_j, \Gamma_j$  are mutually independent sequences. The series (7.1) is a.s. convergent in  $H$  for each  $t$  by lemma 5.1. The random function  $X(\cdot)$  is clearly stationary because  $\Theta_j$  are uniform on  $[-\pi, \pi]$ , but this will also be a simple consequence of the form of the characteristic function which we now compute.

$$\begin{aligned} (7.2) \quad \log_e E e^{i(x, \sum_{k=1}^n r_k X(t_k))} \\ = \log_e E e^{i \sum_{j=1}^{\infty} \left( \sum_{k=1}^n r_k \cos((\Lambda_j, t_k)_0 + \Theta_j) \right) \varepsilon_j(x, X_j) r_j^{-1/\alpha}} \\ (5.1) \quad B(\alpha) E |(x, X_1)|^\alpha E \left| \sum_{k=1}^n r_k \cos((\Lambda_1, t_k)_0 + \Theta_1) \right|^\alpha. \end{aligned}$$

The final term in the right side of (7.2) reduces as follows, with

$$z \triangleq \sum_1^n r_k e^{i(\Lambda_1, t_k)_0},$$

$$\begin{aligned} (7.3) \quad E \left| \sum_1^n r_k \cos((\Lambda_1, t_k)_0 + \Theta_1) \right|^2 &= E 2^{-\alpha} |z e^{i\Theta_1} + \bar{z} e^{-i\Theta_1}|^2 \\ &= E |z|^\alpha 2^{-\alpha} |1 + z^{-1} \bar{z} e^{-2i\Theta_1}|^2 \\ &= E |z|^\alpha 2^{-2\alpha} C(\alpha) \end{aligned}$$

where

$$C(\alpha) = \int_{-\pi}^{\pi} |1 + e^{i\eta}|^\alpha d\eta = \int_{-\pi}^{\pi} |1 + e^{i\psi - 2i\eta}|^\alpha d\eta$$

for all real  $\psi$ . We have therefore proved,

Theorem 7.4. The random function defined by (7.1) converges a.s. in  $H$  for each  $t$ , and has log characteristic function

$$2^{-\alpha} B(\alpha) C(\alpha) E \left| \sum_{k=1}^n r_k e^{i(\Lambda_1, t_k)_0} \right|^\alpha$$

for all  $n \geq 1, r_1, \dots, r_n, t_1, \dots, t_n$ .  $\square$

Corollary 7.5. The random function (7.1) is non-ergodic for each  $\alpha < 2$ .

Proof. By using (7.2) the construction (7.1) remains valid if  $a_j$  are replaced by  $Z_j / (E|Z_1|^\alpha)^{1/\alpha}$ ,  $j \geq 1$ , where  $Z$  is an independent i.i.d. standard normal sequence. Conditional on the sequences  $\Lambda, X, T$ , the

process (7.1) is a.s. stationary Gaussian with discrete spectrum,  
therefore conditionally non-ergodic a.s..



## 8. Operator Stable and Infinitely Divisible Multidimensional Laws.

Infinitely divisible laws, of which the operator stable laws are a special case with particularly interesting structure, are treated in Part II. In brief, this is what happens: A construction of infinitely divisible random vectors is given by  $\Sigma\{X_j H(\Gamma_j, X_j) - \gamma_j\}$ , in which the real function  $H$  is monotone decreasing and positive for each value  $X_j$ , and is determined from the Lévy measure. A construction of full operator stable random vectors in a finite dimensional real vector space  $R^d$  is  $\Sigma\{A(\Gamma_j^{-1})X_j - \gamma_j\}$ , in which the vectors  $\gamma$  are non-stochastic centerings,  $\{A(t) = \exp(B \log t), t > 0\}$  is the group of linear transformations figuring in the definition of operator stability (e.g. Sharpe [9]), and the vectors  $X$  are i.i.d. from a probability measure (a factor of the Lévy measure) on a set of generators of the subgroups induced by  $A$ . The methods of sections 4 and 5 carry over, as will now be indicated. If  $X$  is any i.i.d. sequence in  $R^d$ , and  $X$  is independent of  $\Gamma$ , then  $\forall x \in R^d, n \geq 1$ ,

$$(8.1) \quad \log_e E e^{i(x, \Sigma_1^{K_n} A(\Gamma_j^{-1})X_j)} = \int_{a_n}^{\infty} E(e^{i(x, A(t)X_1)} - 1) \frac{dt}{t^2}.$$

As usual, the symmetric case is simplest. If we examine Sharpe's Theorem 5, we discover that the limit of (8.1) is precisely the form taken by the operator stable in this case, provided we choose for the distribution of  $X_1$  the probability measure figuring in Sharpe's representation of the Lévy measure as a mixture, this measure being placed on (Sharpe's notation) generators  $\theta$  characterized by  $s M_\theta(t^{B\theta} : t > s) = s^{-1}, \forall s > 0$ . Arguing as in section 5, we conclude  $\Sigma_1^{\infty} A(\Gamma_j^{-1})X_j$  converges a.s. in  $R^d$  and has the log-characteristic function which is the limit of (8.1).

9. Priority of P. Lévy.

P. Lévy has anticipated the series constructions of one dimensional stable r.v. with  $\alpha < 2$ . For the case of a positive stable with  $\alpha < 1$ , up to scale and location, this construction is  $\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}$ , with  $\{\Gamma_j, j > 1\}$  being the arrival times of a Poisson process (on  $\mathbb{R}^+$ ) having unit intensity function. Lévy writes the series in the form  $\sum_x U_x$ , where

$$(9.1) \quad \{U_x, x > 0\} \text{ are independent r.v. and}$$

$$(9.2) \quad P(U_x = x) = \frac{\alpha dx}{x^{1+\alpha}} = 1 - P(U_x = 0) .$$

Here is my abstract of the key parts of Lévy's (1935) arguments for the above case:

$$(9.3) \quad \left[ \int_{x_0}^{\infty} \frac{\alpha dx}{x^{1+\alpha}} = x_0^{-\alpha} < \infty \right] \Rightarrow [\{U_x : U_x \neq 0, x > x_0\} \text{ is finite for } x_0 > 0] ,$$

and also,

$$(9.4) \quad \left[ \int_0^{x_0} \frac{\alpha dx}{x^\alpha} = \frac{\alpha}{1-\alpha} x_0^{(1-\alpha)} + 0 \right] \Rightarrow [E \sum_{x < x_0} U_x + 0] (\text{as } x_0 \downarrow 0) .$$

Therefore, for arbitrary  $c_1, c_2 > 0$  (defining  $c_3^\alpha = c_1^\alpha + c_2^\alpha$  and taking independent copies),

$$\begin{aligned}
(9.5) \quad c_1 \sum_x U_x^{(1)} + c_2 \sum_x U_x^{(2)} &\stackrel{\text{Dist.}}{=} \sum_x Y_x^{(1)} + \sum_x Y_x^{(2)} \\
&\stackrel{\text{Dist.}}{=} \sum_x Y_x^{(x)} \\
&\stackrel{\text{Dist.}}{=} c_3 \sum_x U_x \quad (\Rightarrow \text{stable}),
\end{aligned}$$

where  $\{Y_x^{(k)}, x > 0\}$  have respective intensities  $c_k^2 \propto dx/x^{1+\alpha}$ ,  $k=1,2,3$ , and are independent for  $k=1,2$ .

The above arguments do yield a proof of the representation if we apply them to the independent sub-sums  $\{U_x: x \in [b_{n+1}, b_n]\}$ ,  $n \geq 1$ , where  $b_n^{-\alpha} = \log n$ . This is essentially the argument of Ferguson-Klass (1972). The particular choice of  $b_n$ ,  $n \geq 1$  is one which ensures that eventually as  $n \rightarrow \infty$  each sub-sum contains at most one summand, so it really is (almost) as though one could add independent  $U_x$  one at a time toward  $x \rightarrow 0$ . A quite different justification is to interpret  $\sum_x U_x$  as a generalized process driven by "white noise"  $\{U_x, x > 0\}$ .

Lévy's observations are easily overlooked. Ferguson-Klass, Vervaat (1979), LePage-Woodroffe-Zinn (1979) (in manuscript form), rediscover the Lévy construction as byproducts of the following independent pursuits respectively: (F-K) - representing the positive non-Gaussian part of an independent increments random function as the sum of its ordered jumps. (V) - examining a shot-noise associated with the asymptotic behavior of the solution of a stochastic difference equation as time is increased, (L-W-Z) - studying the limit behavior of the normalized order statistics

from a distribution attracted to a stable. Resnick (1976) reconciles the Ferguson-Klass construction with the Ito representation, meaning by the latter Ito's generalization of Lévy's stochastic integral construction by a Poisson random measure.

Acknowledgement. The estimate (4.6) was suggested by J. Zinn. Helpful discussions were also held with V. Mandrekar, M. Woodroffe, and M. Steele.

## References

- [1] Ferguson, T. and Klass, M. (1972). A representation of independent increments processes without Gaussian components. Ann. Math. Statist., 43, 5, 1634-1643.
- [2] Hoffman-Jørgensen, J. (1977). Probability in B-spaces. Lecture note series, No. 45, Aarhus Universitet.
- [3] Kuelbs, J. (1973). A representation theorem for symmetric stable processes and stable measures on  $H$ . Z.W. Verw. Geb. 26, 259-271.
- [4] LePage, R., Woodroffe, M., and Zinn, J. (1980). Convergence to a stable limit distribution via order statistics.
- [5] Lévy, P. (1935). Propriétés asymptotiques des sommes de variables aléatoires indépendantes ou enchainées. Journ. de Math., tome XIV.-Fasc. IV.
- [6] Logan, B.F., Mallows, C.F., Rice, S.O., and Shepp, L.A. (1973). Limit distributions of self-normalized sums. Ann. Prob., 1, 5, 788-809.
- [7] Resnick, S. (1976). An extremal decomposition of a process with stationary, independent increments. Tech. Rep. No. 79, Dept. of Statistics, Stanford University, Stanford, California.
- [8] Schilder, M. (1970). Some structure theorems for symmetric stable law. Ann. Math. Statist., 41, 412-421.
- [9] Sharpe, M. (1969). Operator-stable probability distributions on vector groups. Transactions Amer. Math. Soc., 135, 51-65.
- [10] Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob., 11, 750-783.

END  
DATE

FILMED

MARCH

1988

DTIC